

Invariant critical sets of conserved quantities

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Abstract

For a dynamical system we will construct various invariant sets starting from its conserved quantities. We will give conditions under which certain solutions of a nonlinear system are also solutions for a simpler dynamical system, for example when they are solutions for a linear dynamical system. We will apply these results to the example of Toda lattice.

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1 Introduction.

A particle moving in the Newtonian gravitational field is known as the Kepler problem. The equations of motion are

$$\ddot{x} = -\frac{x}{\|x\|^3},$$

where $x \in \mathbb{R}^3 \setminus \{0\}$ is the position vector.

It is well known that the motions for the Kepler problem are planar. If we consider the plane Ox_1x_2 , the equations of motion becomes

$$\begin{cases} \dot{x}_1 = y_1 \\ \dot{x}_2 = y_2 \\ \dot{y}_1 = -\frac{x_1}{(x_1^2+x_2^2)^{3/2}} \\ \dot{y}_2 = -\frac{x_2}{(x_1^2+x_2^2)^{3/2}} \end{cases}$$

These equations are Hamiltonian with the standard symplectic form on \mathbb{R}^4 and the Hamiltonian function $H = \frac{1}{2}(y_1^2 + y_2^2) - \frac{1}{\sqrt{x_1^2+x_2^2}}$. From Kepler's second law we have another conserved quantity given by $A = x_1y_2 - x_2y_1$.

For $a > 0$, consider the following conserved quantity $K = H + \frac{1}{a^3}A$. After a straightforward computation we will obtain that $\nabla K = 0$ if and only if $y_1 = \frac{1}{a^3}x_2$, and $y_2 = -\frac{1}{a^3}x_1$ and $\|(x_1, x_2)\| = a^2$. Equivalently, the set $\{\nabla K = 0\}$ is equal with the set $\{(x_1, x_2, y_1, y_2) \mid (x_1, x_2) \cdot (y_1, y_2) = 0, \text{ and } \|(x_1, x_2)\| = a^2, \text{ and } \|(y_1, y_2)\| = \frac{1}{a} \text{ and } \text{sgn}(y_1) = \text{sgn}(x_2)\}$ which is invariant under the dynamics and is filled with solutions that represent uniform circular motions moving clockwise. These particular motions are also solutions for the linear Hamiltonian system

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \\ \dot{y}_1 = -y_2 \\ \dot{y}_2 = y_1 \end{cases},$$

where the Hamiltonian function is A .

Given the above analysis, we can raise at least two questions. Is it true that for a dynamical system that admits conserved quantities, the set of points where the gradients of these conserved quantities are zero, is an invariant set? When do two Hamiltonian systems have common solutions? In what follows, we will give an answer to these questions.

For the first question, the answer is positive and it is given in section two where we will also discuss various generalizations of this answer. In section three, we will present the conditions under which we can give an answer to the second question. In section four we will illustrate these results for the example of Toda lattice. Detailed computations are presented in the Appendix.

2 Invariant sets

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^q , $q \geq 1$ function which generates the differential equation

$$\dot{x} = f(x). \quad (2.1)$$

We suppose that equation (2.1) admits a C^q vectorial conserved quantity $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^k$ with $k \leq n$. We denote by F_1, \dots, F_k the components of \mathbf{F} . For $s \in \{0, \dots, k\}$ we introduce the sets:

$$M_{(s)}^F = \{x \in \mathbb{R}^n \mid \text{rank} \nabla \mathbf{F}(x) = s\} \quad (2.2)$$

where $\nabla \mathbf{F}(x)$ is the Jacobian matrix

$$\nabla \mathbf{F}(x) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(x) & \dots & \frac{\partial F_1}{\partial x_n}(x) \\ \dots & \dots & \dots \\ \frac{\partial F_k}{\partial x_1}(x) & \dots & \frac{\partial F_k}{\partial x_n}(x) \end{pmatrix}. \quad (2.3)$$

For $r \in \{1, \dots, q\}$ we introduce the sets:

$$N_{(r)}^F = \{x \in \mathbb{R}^n \mid \partial^\alpha F_i(x) = 0, \forall i \in \{1, \dots, k\}, \forall \alpha \in \{1, \dots, n\}^l, \forall l \leq r\} \quad (2.4)$$

where we note $\partial^\alpha F_i = \frac{\partial^l F_i}{\partial x_{\alpha_1} \dots \partial x_{\alpha_l}}$ if $\alpha = (\alpha_1, \dots, \alpha_l)$.

Remark 2.1 *We observe that*

$$M_{(0)}^F = N_{(1)}^F = \{x \in \mathbb{R}^n \mid \frac{\partial F_i}{\partial x_j}(x) = 0, i \in \{1, \dots, k\}, j \in \{1, \dots, n\}\}. \quad (2.5)$$

The set $\{M_{(s)}^F\}$ with $s \in \{0, \dots, k\}$ is a partition of \mathbb{R}^n . A critical point of \mathbf{F} is a point in \mathbb{R}^n at which the rank of the matrix $\nabla \mathbf{F}(x)$ is less than the maximum rank. A critical value is the image under \mathbf{F} of a critical point. The set of critical points of \mathbf{F} is

$$M_c^F = \cup_{s=0}^{k-1} M_{(s)}^F. \quad (2.6)$$

Using Sard's Theorem (see [9]) we have that the set of critical values $\mathbf{F}(M_c^F)$ is of k -dimensional measure zero providing that $q \geq n - k + 1$.

Remark 2.2 *We also have the obvious inclusions $N_{(q)}^F \subseteq N_{(q-1)}^F \subseteq \dots \subseteq N_{(1)}^F$.*

Theorem 2.3 *The sets $M_{(s)}^F$ are invariant under the dynamics generated by the differential equation (2.1).*

Proof. Because \mathbf{F} is a conserved quantity, we have

$$\mathbf{F}(\Phi_t(x)) = \mathbf{F}(x),$$

where $\Phi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the flow generated by (2.1). Differentiating, we have

$$\nabla \mathbf{F}(\Phi_t(x)) \nabla \Phi_t(x) = \nabla \mathbf{F}(x). \quad (2.7)$$

As $\nabla\Phi_t(x)$ is an invertible matrix for any $x \in \mathbb{R}^n$ which is not an equilibrium point for (2.1), (see [1]), we have that

$$\text{rank}\nabla\mathbf{F}(\Phi_t(x)) = \text{rank}\nabla\mathbf{F}(x),$$

which implies the stated result. ■

As a consequence we will obtain the following well known result which was applied for studying invariant sets of various mechanical systems, see for example [6].

Corollary 2.4 *The set of critical points of \mathbf{F} is an invariant set of the dynamics generated by the differential equation (2.1).*

Theorem 2.5 *The sets $N_{(r)}^F$ are invariant sets for the dynamics generated by the differential equation (2.1).*

Proof. Let $i \in \{1, \dots, k\}$, $l \in \{1, \dots, q\}$ and $\alpha = (\alpha_1, \dots, \alpha_l) \in \{1, \dots, n\}^l$. We will prove by mathematical induction that

$$\partial^\alpha F_i(x) = \sum_{\beta_1, \dots, \beta_l=1}^n (\nabla\Phi_t)_{\alpha_1\beta_1}(x) \dots (\nabla\Phi_t)_{\alpha_l\beta_l}(x) \partial^\beta F_i(\Phi_t(x)) + S_{i,\alpha}(t, x), \quad (2.8)$$

where $\beta = (\beta_1, \dots, \beta_l)$ and $S_{i,\alpha}(t, x)$ is a sum with the property: "all the terms contain a factor of the form $\partial^\gamma F_i(\Phi_t(x))$ with $|\gamma| < l$ ".

We will prove this result by mathematical induction with respect to l . Componentwise the relation (2.7) implies our result for $l = 1$.

Let $\alpha' = (\alpha_1, \dots, \alpha_l, \alpha_{l+1}) \in \{1, \dots, n\}^{l+1}$ where $\alpha = (\alpha_1, \dots, \alpha_l) \in \{1, \dots, n\}^l$. Using the induction hypothesis we have:

$$\partial^{\alpha'} F_i(x) = \frac{\partial}{\partial x_{\alpha_{l+1}}} \left(\sum_{\beta_1, \dots, \beta_l=1}^n (\nabla\Phi_t)_{\alpha_1\beta_1}(x) \dots (\nabla\Phi_t)_{\alpha_l\beta_l}(x) \partial^\beta F_i(\Phi_t(x)) + \frac{\partial}{\partial x_{\alpha_{l+1}}} (S_{i,\alpha}(t, x)) \right).$$

By a straightforward computation we obtain

$$\begin{aligned} & \frac{\partial}{\partial x_{\alpha_{l+1}}} \left(\sum_{\beta_1, \dots, \beta_l=1}^n (\nabla\Phi_t)_{\alpha_1\beta_1}(x) \dots (\nabla\Phi_t)_{\alpha_l\beta_l}(x) \partial^\beta F_i(\Phi_t(x)) \right) = \\ &= \sum_{\beta_1, \dots, \beta_l=1}^n (\nabla\Phi_t)_{\alpha_1\beta_1}(x) \dots (\nabla\Phi_t)_{\alpha_l\beta_l}(x) \frac{\partial}{\partial x_{\alpha_{l+1}}} \partial^\beta F_i(\Phi_t(x)) + \\ &+ \sum_{\beta_1, \dots, \beta_l=1}^n \frac{\partial}{\partial x_{\alpha_{l+1}}} ((\nabla\Phi_t)_{\alpha_1\beta_1}(x) \dots (\nabla\Phi_t)_{\alpha_l\beta_l}(x)) \partial^\beta F_i(\Phi_t(x)) = \\ &= \sum_{\beta_1, \dots, \beta_l, \beta_{l+1}=1}^n (\nabla\Phi_t)_{\alpha_1\beta_1}(x) \dots (\nabla\Phi_t)_{\alpha_l\beta_l}(x) (\nabla\Phi_t)_{\alpha_{l+1}\beta_{l+1}}(x) \partial^{\beta'} F_i(\Phi_t(x)) + \\ &+ \sum_{\beta_1, \dots, \beta_l=1}^n \frac{\partial}{\partial x_{\alpha_{l+1}}} ((\nabla\Phi_t)_{\alpha_1\beta_1}(x) \dots (\nabla\Phi_t)_{\alpha_l\beta_l}(x)) \partial^\beta F_i(\Phi_t(x)). \end{aligned}$$

We will note that

$$S_{i,\alpha'}(t, x) = \sum_{\beta_1, \dots, \beta_l=1}^n \frac{\partial}{\partial x_{\alpha_{l+1}}} ((\nabla\Phi_t)_{\alpha_1\beta_1}(x) \dots (\nabla\Phi_t)_{\alpha_l\beta_l}(x)) \partial^\beta F_i(\Phi_t(x)) + \frac{\partial}{\partial x_{\alpha_{l+1}}} (S_{i,\alpha}(t, x)).$$

All the terms of $S_{i,\alpha'}(t, x)$ contain a factor of the form $\partial^\gamma F_i(\Phi_t(x))$ with $|\gamma| < l + 1$, which had to be proved.

Let $\beta = (\beta_1, \dots, \beta_l) \in \{1, \dots, n\}^l$ and $\nabla\Phi_t^{-1}(x)$ be the inverse matrix of $\nabla\Phi_t(x)$, where x is not an equilibrium point for (2.1). Consequently, we have

$$\partial^\beta F_i(\Phi_t(x)) = \sum_{\alpha_1, \dots, \alpha_l=1}^n (\nabla\Phi_t)_{\beta_1\alpha_1}^{-1}(x) \dots (\nabla\Phi_t)_{\beta_l\alpha_l}^{-1}(x) [\partial^\alpha F_i(x) - S_{i,\alpha}(t, x)]. \quad (2.9)$$

Also, we will prove by mathematical induction that the sets $N_{(j)}^F$ are invariant under the dynamics generated by the differential equation (2.1). For $j = 1$ we have $N_{(1)}^F = M_{(0)}^F$, which is an invariant set (see Theorem 2.3). We suppose that for all $j \leq l$ the sets $N_{(j)}^F$ are invariant. Let $x \in N_{(l+1)}^F$ be arbitrary chosen. Using (2.9) for $l+1$ -order of derivation and the induction hypothesis, we deduce that $\Phi_t(x) \in N_{(l+1)}^F$, $\forall t$. Summing up, $N_{(j)}^F$ are invariant sets for all $j \in \{1, \dots, q\}$. ■

3 Finding solutions using simpler dynamics

Let $F, G : \mathbb{R}^n \rightarrow \mathbb{R}$ be C^q functions and the differential equations:

$$\dot{x} = f(x, \nabla F(x)) \quad (3.1)$$

and

$$\dot{x} = f(x, \nabla G(x)) \quad (3.2)$$

where $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ is a C^q vectorial function.

For $x \in \mathbb{R}^n$, we denote with $\Phi_t^F(x)$ and $\Phi_t^G(x)$ the solutions of (3.1) and (3.2) with the initial conditions $\Phi_0^F(x) = x$ and $\Phi_0^G(x) = x$. We will introduce the following set

$$E_1 = \{x \in \mathbb{R}^n \mid \nabla F(x) = \nabla G(x)\}.$$

Theorem 3.1 *If $F - G$ is a conserved quantity for (3.1) and $x \in E_1$, then for all t we have*

$$\Phi_t^F(x) = \Phi_t^G(x).$$

Proof. For $L = F - G$ we have the equality $E_1 = M_{(0)}^L$. By Theorem (2.3) the set E_1 is invariant under the dynamics of (3.1). For $x \in E_1$ we have $\nabla F(\Phi_t^F(x)) = \nabla G(\Phi_t^F(x))$ for all t and consequently

$$\frac{d}{dt}\Phi_t^F(x) = f(\Phi_t^F(x), \nabla F(\Phi_t^F(x))) = f(\Phi_t^F(x), \nabla G(\Phi_t^F(x))).$$

The above equality shows that $\Phi_t^F(x)$ is also a solution for (3.2). Given the uniqueness of the solutions, we obtain the desired equality. ■

Next we will discuss a particular case of the result presented above. For this we will take $f(x, y) = h(x) + g(y)$ and $F \equiv 0$. Thus we have the two dynamics

$$\dot{x} = h(x) \quad (3.3)$$

and the perturbed dynamics

$$\dot{x} = h(x) + g(\nabla G(x)). \quad (3.4)$$

We denote by Φ_t^h the flux for (3.3).

Corollary 3.2 *If G is a conserved quantity for (3.3), then for all initial conditions $x \in \{x \in \mathbb{R}^n \mid \nabla G(x) = 0\}$ we have*

$$\Phi_t^h(x) = \Phi_t^G(x).$$

Another particular case is when the function f in (3.1) and (3.2) verifies the equality

$$\langle f(x, y), y \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidian product on \mathbb{R}^n . In this case F is a conserved quantity for (3.1) and G is a conserved quantity for (3.2). The above equality is verified for the case when

$$f(x, y) = \Pi(x)y,$$

where $\Pi(x)$ is an antisymmetric matrix. This is true for all almost Poisson manifolds (see [8]). In this situation the differential equations (3.1) and (3.2) become

$$\dot{x} = \Pi(x)\nabla F(x) \tag{3.5}$$

and

$$\dot{x} = \Pi(x)\nabla G(x). \tag{3.6}$$

We observe that F is a conserved quantity for (3.6) if and only if G is a conserved quantity for (3.5).

Corollary 3.3 *If G is a conserved quantity for (3.5), then for the initial conditions in $\{x \in \mathbb{R}^n \mid \nabla F(x) = \nabla G(x)\}$ we have $\Phi_t^F(x) = \Phi_t^G(x)$.*

A particular case of Corollary 3.3 is when we have a symplectic manifold with G being a quadratic function. In this case, the solutions of the Hamiltonian vector field X_F starting in $\{x \in \mathbb{R}^n \mid \nabla F(x) = \nabla G(x)\}$ are also the solutions of the linear Hamiltonian system X_G .

Analogous results are valid for the more general case of the vector valued conserved quantities and when the right side of equations (3.1) and (3.2) depends on higher order derivatives.

Firstly, we will introduce some notations. Let $\mathbf{F}, \mathbf{G} : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be C^q vectorial functions with $q \geq 1$ and also $k \leq n$. If F_1, \dots, F_k are the components of \mathbf{F} we denote with

$$\Delta^1 \mathbf{F}(x) = \left(\frac{\partial F_1}{\partial x_1}(x), \dots, \frac{\partial F_1}{\partial x_n}(x), \frac{\partial F_2}{\partial x_1}(x), \dots, \frac{\partial F_2}{\partial x_n}(x), \dots, \frac{\partial F_k}{\partial x_1}(x), \dots, \frac{\partial F_k}{\partial x_n}(x) \right) \in \mathbb{R}^{kn}.$$

and

$$\Delta^r \mathbf{F}(x) = (\dots, \partial^\alpha F_i(x), \dots) \in \mathbb{R}^{kn^r},$$

where $r \in \{1, \dots, q\}$, $i \in \{1, \dots, k\}$, $\alpha \in \{1, \dots, n\}^r$ ($|\alpha| = r$) and the components appear in the lexicographical order for (i, α) in \mathbb{N}^{r+1} .

For a fix $r \in \{1, \dots, q\}$, we will consider, as before, the two differential equations

$$\dot{x} = f_r(x, \Delta^1 \mathbf{F}(x), \dots, \Delta^r \mathbf{F}(x)) \tag{3.7}$$

and

$$\dot{x} = f_r(x, \Delta^1 \mathbf{G}(x), \dots, \Delta^r \mathbf{G}(x)). \tag{3.8}$$

Let $x \in \mathbb{R}^n$, we denote with $\Phi_t^{\mathbf{F}}(x)$ and $\Phi_t^{\mathbf{G}}(x)$ the solutions of (3.7) and (3.8) with initial conditions $\Phi_0^{\mathbf{F}}(x) = x$ and $\Phi_0^{\mathbf{G}}(x) = x$. We will introduce the following set

$$E_r = \{x \in \mathbb{R}^n \mid \partial^\alpha \mathbf{F}(x) = \partial^\alpha \mathbf{G}(x), \forall |\alpha| \leq r\}.$$

Theorem 3.4 *If $\mathbf{F} - \mathbf{G}$ is a conserved quantity for (3.7) and $x \in E_r$ then for all t we have*

$$\Phi_t^{\mathbf{F}}(x) = \Phi_t^{\mathbf{G}}(x).$$

Also the obvious extension of Corollary 3.2 takes place.

4 Invariant sets for Toda lattices

The Toda lattice describes the one-dimensional motions of a chain of particles with nearest neighbor interactions. For a chain of particles with the equal masses m , Morikazu Toda came up with the choice of the interaction potential

$$V(r) = e^{-r} + r - 1.$$

The system of motion reads explicitly

$$m\ddot{x}_i = e^{-(x_i - x_{i-1})} - e^{-(x_{i+1} - x_i)}, \quad i \in \mathbb{Z}. \quad (4.1)$$

This second order differential system is equivalent with the first order differential system

$$\begin{cases} \dot{x}_i = u_i \\ m\dot{u}_i = e^{-(x_i - x_{i-1})} - e^{-(x_{i+1} - x_i)}, \quad i \in \mathbb{Z}, \end{cases} \quad (4.2)$$

where u_i is the velocity of the particle i .

An equilibrium of the Toda lattice has the form

$$x_i = x_0 + \lambda i, \quad u_i = 0, \quad \lambda \in \mathbb{R}^*, \quad x_0 \in \mathbb{R}, \quad i \in \mathbb{Z}.$$

Let an equilibrium of the Toda lattice and $y_i = x_i - x_0 - \lambda i$ the displacement of the i particle from its equilibrium position. The system in the variables y_i and u_i is

$$\begin{cases} \dot{y}_i = u_i \\ \dot{u}_i = \frac{e^{-\lambda}}{m} (e^{-(y_i - y_{i-1})} - e^{-(y_{i+1} - y_i)}), \quad i \in \mathbb{Z} \end{cases} \quad (4.3)$$

Let us define

$$X_i := \frac{e^{-\lambda}}{m} e^{-(y_{i+1} - y_i)}, \quad (4.4)$$

then the equations of motion become

$$\begin{cases} \dot{X}_i = X_i(u_i - u_{i+1}) \\ \dot{u}_i = X_{i-1} - X_i, \quad i \in \mathbb{Z}. \end{cases} \quad (4.5)$$

The following particular cases are interesting:

1. the case of a periodic lattice, $X_{i+n} = X_i \quad \forall i \in \mathbb{Z}$,
2. the case of a non-periodic lattice with the boundary conditions $X_0 = 0$ (correspond to formally setting $y_0 = -\infty$) and $X_n = 0$ (correspond to formally setting $y_{n+1} = \infty$).

In both cases we investigate the motions of the particles 1 to n ($n \in \mathbb{N}^*$).

4.1 The case of a periodic lattice

In this case *M. Hénon* proves in [5] that the following expressions are scalar conserved quantities

$$I_m = \sum u_{i_1} \dots u_{i_k} (-X_{j_1}) \dots (-X_{j_l}), \quad (4.6)$$

where $m \in \{1, \dots, n\}$ and the summation are extended to all terms which satisfy the following conditions:

- (i) the indices $i_1, \dots, i_k, j_1, j_1 + 1, \dots, j_l, j_l + 1$, which appear in the term (either explicitly, or implicitly through a factor X_j) are all different (modulo n);
- (ii) the number of these indices is m , i.e. $k + 2l = m$. Two terms differing only in the order of factors are not considered different, and therefore only one of them appears in the sum.

In [4], Flaschka has proved that the above functions are conserved quantities using a Lax formulation. This was generalized to arbitrary Lie algebras by Adler [2] and Kostant [7].

The first three scalar conserved quantities, depending on the variables $(X_1, \dots, X_n, u_1, \dots, u_n)$, are

$$I_1 = \sum_{1 \leq i \leq n} u_i \quad (4.7)$$

$$I_2 = \sum_{1 \leq i_1 < i_2 \leq n} u_{i_1} u_{i_2} - \sum_{1 \leq j \leq n} X_j \quad (4.8)$$

$$I_3 = \sum_{1 \leq i_1 < i_2 < i_3 \leq n} u_{i_1} u_{i_2} u_{i_3} - \sum_{1 \leq i, j \leq n, j \neq i, j \neq i-1 \pmod{n}} u_i X_j, \quad (X_0 = X_n). \quad (4.9)$$

We will introduce the vectorial conserved quantities,

$$\mathbb{I}_{12} = (I_1, I_2), \quad \mathbb{I}_{13} = (I_1, I_3), \quad \mathbb{I}_{23} = (I_2, I_3), \quad \mathbb{I}_{123} = (I_1, I_2, I_3). \quad (4.10)$$

I. The case n odd

For this case we obtain as invariant sets only subsets of the set of equilibrium points or the empty set. More precisely, $M_{(0)}^{I_1} = M_{(0)}^{I_2} = M_{(0)}^{I_{12}} = M_{(1)}^{I_{12}} = M_{(0)}^{I_{13}} = M_{(0)}^{I_{23}} = M_{(0)}^{I_{123}} = M_{(1)}^{I_{123}} = \emptyset$. The following are subsets of the set of equilibrium points:

$$M_{(0)}^{I_3} = \{(0, \dots, 0, 0, \dots, 0)\},$$

$$M_{(1)}^{I_{13}} = \{(X, \dots, X, 0, \dots, 0) \mid X \in \mathbb{R}\},$$

$$M_{(1)}^{I_{23}} = \{(-\frac{n-1}{2}u^2, \dots, -\frac{n-1}{2}u^2, u, \dots, u) \mid u \in \mathbb{R}\},$$

$$M_{(2)}^{I_{123}} = \{(X, \dots, X, u, \dots, u) \mid X, u \in \mathbb{R}\}.$$

II. The case n even

We have, $M_{(0)}^{I_1} = M_{(0)}^{I_2} = M_{(0)}^{I_{12}} = M_{(1)}^{I_{12}} = M_{(0)}^{I_{13}} = M_{(0)}^{I_{23}} = M_{(0)}^{I_{123}} = M_{(1)}^{I_{123}} = \emptyset$. As nontrivial invariant sets we have the following:

$$M_{(0)}^{I_3} = \{(X_1, X_2, \dots, X_1, X_2, u_1, u_2, \dots, u_1, u_2) \mid X_1 + X_2 = u_1 u_2, u_1 + u_2 = 0\},$$

$$M_{(1)}^{I_{13}} = \{(X_1, X_2, \dots, X_1, X_2, u_1, u_2, \dots, u_1, u_2) \mid u_1 + u_2 = 0\},$$

$$M_{(1)}^{I_{23}} = \{(X_1, X_2, \dots, X_1, X_2, u_1, u_2, \dots, u_1, u_2) \mid X_1 + X_2 = -\frac{n}{4}(u_1 + u_2)^2 + u_1 u_2\},$$

$$M_{(2)}^{I_{123}} = \{(X_1, X_2, \dots, X_1, X_2, u_1, u_2, \dots, u_1, u_2) \mid X_1, X_2, u_1, u_2 \in \mathbb{R}\}.$$

We obtain $M_{(2)}^{I_{123}}$ as the largest invariant set and the restricted dynamics is the dynamics of two particles

$$\begin{cases} \dot{X}_1 = X_1(u_1 - u_2) \\ \dot{X}_2 = X_2(u_2 - u_1) \\ \dot{u}_1 = X_2 - X_1 \\ \dot{u}_2 = X_1 - X_2 \end{cases}$$

On the invariant set $M_{(1)}^{I_{23}}$ we have the above dynamics subject to $X_1 + X_2 = -\frac{n}{4}(u_1 + u_2)^2 + u_1 u_2$. On the invariant set $M_{(1)}^{I_{13}}$ we have the above dynamics subject to $u_1 + u_2 = 0$ and on the invariant set $M_{(0)}^{I_3}$ we have the above dynamics subject to $u_1 + u_2 = 0$ and $X_1 + X_2 = u_1 u_2$.

For the sets $M_{(0)}^{I_3}$ and $M_{(1)}^{I_{23}}$ the variables X_i have to take also negative values which from a mathematical point of view is correct and can be the solutions for the system (4.5). As the mechanical system is given by (4.3) and we do the change of variables (4.4), the variables X_i have to be strictly positive in order to have a physical meaning. Consequently, from a mechanical point of view only the sets $M_{(1)}^{I_{13}}$ and $M_{(2)}^{I_{123}}$ are meaningful. The computations can be found in the Appendix.

4.2 The case of non-periodic lattice

It is known that if we have the matrices

$$L = \begin{pmatrix} u_1 & X_1 & 0 & \dots & 0 \\ 1 & u_2 & X_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & u_{n-1} & X_{n-1} \\ 0 & \dots & 0 & 1 & u_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -X_1 & 0 & \dots & 0 \\ 0 & 0 & -X_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & -X_{n-1} \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

then the system (4.5), in this case, has the Lax form

$$\dot{L} = [B, L], \quad (4.11)$$

where $[B, L] = BL - LB$. Using the Flaschka theorem (see [4]) we have the following scalar conserved quantities depending on the variables $(X_1, \dots, X_{n-1}, u_1, \dots, u_n)$

$$F_k = \frac{1}{k} \text{tr}(L^k), \quad k \in \{1, \dots, n\}. \quad (4.12)$$

For $k \in \{1, 2, 3\}$ we have

$$\begin{aligned} F_1 &= \sum_{i=1}^n u_i \\ F_2 &= \sum_{i=1}^n \left(X_i + \frac{u_i^2}{2} \right) \\ F_3 &= \sum_{i=1}^{n-1} X_i (u_i + u_{i+1}) + \frac{1}{3} \sum_{i=1}^n u_i^3. \end{aligned}$$

We will introduce the vectorial conserved quantities,

$$\mathbb{F}_{12} = (F_1, F_2), \quad \mathbb{F}_{13} = (F_1, F_3), \quad \mathbb{F}_{23} = (F_2, F_3), \quad \mathbb{F}_{123} = (F_1, F_2, F_3). \quad (4.13)$$

As before we will distinguish two cases.

I. The case n odd

In this case we obtain as invariant sets only subsets of the set of equilibrium points or the empty set. More precisely, $M_{(0)}^{F_1} = M_{(0)}^{F_2} = M_{(0)}^{\mathbb{F}_{12}} = M_{(1)}^{\mathbb{F}_{12}} = M_{(0)}^{\mathbb{F}_{13}} = M_{(0)}^{\mathbb{F}_{23}} = M_{(0)}^{\mathbb{F}_{123}} = M_{(1)}^{\mathbb{F}_{123}} = \emptyset$. The following are subsets of the set of equilibrium points:

$$\begin{aligned} M_{(0)}^{F_3} &= M_{(1)}^{\mathbb{F}_{13}} = \{(\underbrace{0, \dots, 0}_{n-1}, \underbrace{0, \dots, 0}_n)\}, \\ M_{(1)}^{\mathbb{F}_{23}} &= \{(\underbrace{0, \dots, 0}_{n-1}, \underbrace{u_1, 0, \dots, u_1, 0, u_1}_n) \mid u_1 \in \mathbb{R}\} \cup \{(\underbrace{0, \dots, 0}_{n-1}, \underbrace{0, u_2, \dots, 0, u_2, 0}_n) \mid u_2 \in \mathbb{R}\}, \\ M_{(2)}^{\mathbb{F}_{123}} &= \{(\underbrace{0, \dots, 0}_{n-1}, \underbrace{u_1, u_2, \dots, u_1}_n) \mid u_1, u_2 \in \mathbb{R}\}. \end{aligned}$$

II. The case n even

We have, $M_{(0)}^{F_1} = M_{(0)}^{F_2} = M_{(0)}^{\mathbb{F}_{12}} = M_{(1)}^{\mathbb{F}_{12}} = M_{(0)}^{\mathbb{F}_{13}} = M_{(0)}^{\mathbb{F}_{23}} = M_{(0)}^{\mathbb{F}_{123}} = M_{(1)}^{\mathbb{F}_{123}} = \emptyset$. As nontrivial invariant sets we have the following:

$$M_{(0)}^{F_3} = \{(\underbrace{X, 0, \dots, X, 0, X}_{n-1}, \underbrace{u_1, u_2, \dots, u_1, u_2}_n) \mid u_1 + u_2 = 0, X = u_1 u_2\},$$

$$\begin{aligned}
M_{(1)}^{\mathbb{F}_{13}} &= \{(X, 0, \dots, X, 0, X, \underbrace{u_1, u_2, \dots, u_1, u_2}_n \mid u_1 + u_2 = 0\}, \\
M_{(1)}^{\mathbb{F}_{23}} &= \{(X, 0, \dots, X, 0, X, \underbrace{u_1, u_2, \dots, u_1, u_2}_n \mid X = u_1 u_2\}, \\
M_{(2)}^{\mathbb{F}_{123}} &= \{(X, 0, \dots, X, 0, X, \underbrace{u_1, u_2, \dots, u_1, u_2}_n \mid X, u_1, u_2 \in \mathbb{R}\}.
\end{aligned}$$

We obtain $M_{(2)}^{\mathbb{F}_{123}}$ as the largest invariant set and the restricted dynamics is given by

$$\begin{cases} \dot{X} = X(u_1 - u_2) \\ \dot{u}_1 = -X \\ \dot{u}_2 = X \end{cases}$$

On the invariant set $M_{(1)}^{\mathbb{F}_{23}}$ we have the above dynamics subject to $X = u_1 u_2$. On the invariant set $M_{(1)}^{\mathbb{F}_{13}}$ we have the above dynamics subject to $u_1 + u_2 = 0$ and on the invariant set $M_{(0)}^{\mathbb{F}_3}$ we have the above dynamics subject to $u_1 + u_2 = 0$ and $X = u_1 u_2$.

For the set $M_{(2)}^{\mathbb{F}_{123}}$ the variables X_i with i even are all equal with zero which from a mathematical point of view is correct. As before, the mechanical system is given by (4.3) and we do the change of variables (4.4). Consequently, the variables X_i have to be strictly positive in order to have a physical meaning. The computations can be found in the Appendix.

5 Appendix

The computations for the case of periodic lattice.

We will make the notations

$$U = \sum_{1 \leq i \leq n} u_i, \quad V = \sum_{1 \leq i_1 < i_2 \leq n} u_{i_1} u_{i_2}, \quad Y = \sum_{1 \leq j \leq n} X_j. \quad (5.1)$$

We observe that

$$2V = U^2 - \sum_{i=1}^n u_i^2. \quad (5.2)$$

With these notations we have the following,

$$\nabla I_1 = (0, \dots, 0, 1, \dots, 1) \quad (5.3)$$

$$\nabla I_2 = (-1, \dots, -1, \underbrace{U - u_1}_{n+1}, \dots, \underbrace{U - u_k}_{n+k}, \dots, \underbrace{U - u_n}_{2n}) \quad (5.4)$$

$$\nabla I_3 = (\dots, \underbrace{-(U - u_k - u_{k+1})}_k, \dots, \underbrace{V - u_k(U - u_k) - (Y - X_{k-1} - X_k)}_{n+k}, \dots). \quad (5.5)$$

The study of $M_{(0)}^{I_3}$.

The elements of $M_{(0)}^{I_3}$ are the solutions of the algebraic system

$$\begin{cases} U - u_k - u_{k+1} = 0, \\ V - u_k(U - u_k) - (Y - X_{k-1} - X_k) = 0, \quad \forall k \in \{1, \dots, n\}. \end{cases} \quad (5.6)$$

Adding the first n equations, we obtain $U = 0$ and $u_1 + u_2 = u_2 + u_3 = \dots = u_{n-1} + u_n = u_n + u_1$. We deduce the following results:

The case $n \in 2\mathbb{N} + 1$. In this situation we have $u_1 = u_2 = \dots = u_n = 0$. Adding the last n equations of (5.6) we obtain $Y = 0$ and $X_1 + X_2 = X_2 + X_3 = \dots = X_{n-1} + X_n = X_n + X_1$. It implies that $X_1 = \dots = X_n = 0$ and consequently

$$M_{(0)}^{I_3} = \{(0, \dots, 0, 0, \dots, 0)\}.$$

The case $n \in 2\mathbb{N}$. For this situation $u_i = (-1)^{i+1}u$, $u \in \mathbb{R}$. In this case, using (5.2), we have $V = -\frac{n}{2}u^2$. The last n equations of (5.6) become

$$-Y + X_{k-1} + X_k = \left(\frac{n}{2} - 1\right)u^2 \quad \forall k \in \{1, \dots, n\}.$$

Making the addition we have $Y = -\frac{n}{2}u^2$ and consequently $X_1 + X_2 = X_2 + X_3 = \dots = X_{n-1} + X_n = X_n + X_1 = -u^2$ which implies $X_1 = X_3 = \dots = X_{n-1}$, and $X_2 = X_4 = \dots = X_{2n}$. In this case we have

$$M_{(0)}^{I_3} = \{(X_1, X_2, \dots, X_1, X_2, u_1, u_2, \dots, u_1, u_2) \mid X_1 + X_2 = u_1 u_2, u_1 + u_2 = 0\}.$$

The study of $M_{(0)}^{\mathbb{I}_{ij}}$, $M_{(1)}^{\mathbb{I}_{ij}}$ **with** $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$.

A point $(X_1, \dots, X_n, u_1, \dots, u_n) \in M_{(1)}^{\mathbb{I}_{13}}$ if and only if we have, for all $k, q \in \{1, \dots, n\}$,

$$\begin{cases} U - u_k - u_{k+1} = 0 \\ V - u_k(U - u_k) - (Y - X_{k-1} - X_k) = V - u_q(U - u_q) - (Y - X_{q-1} - X_q). \end{cases} \quad (5.7)$$

Adding the first n equations we obtain $U = 0$, $u_1 + u_2 = u_2 + u_3 = \dots = u_{n-1} + u_n = u_n + u_1$ and consequently $u_i = (-1)^{i+1}u$, $u \in \mathbb{R}$. The last n equations become $X_1 + X_2 = X_2 + X_3 = \dots = X_{n-1} + X_n = X_n + X_1$.

The case $n \in 2\mathbb{N} + 1$. For this case note that $u_1 = u_2 = \dots = u_n = 0$ and $X_1 = \dots = X_n = X \in \mathbb{R}$. In this case we have

$$M_{(1)}^{\mathbb{I}_{13}} = \{(X, \dots, X, 0, \dots, 0) \mid X \in \mathbb{R}\}.$$

The case $n \in 2\mathbb{N}$. It is easy to see that $X_1 = X_3 = \dots = X_{n-1}$, $X_2 = X_4 = \dots = X_n$. In this case we have

$$M_{(1)}^{\mathbb{I}_{13}} = \{(X_1, X_2, \dots, X_1, X_2, u_1, u_2, \dots, u_1, u_2) \mid u_1 + u_2 = 0\}.$$

A point belongs to the set $M_{(1)}^{\mathbb{I}_{23}}$ if and only if

$$\det(A_{kq}) = 0, \quad \det(B_{kq}) = 0, \quad \det(C_{kq}) = 0, \quad \forall k, q \in \{1, \dots, n\}, \quad (5.8)$$

where

$$\begin{aligned} A_{kq} &= \begin{pmatrix} -1 & -1 \\ -(U - u_k - u_{k+1}) & -(U - u_q - u_{q+1}) \end{pmatrix} \\ B_{kq} &= \begin{pmatrix} -1 & U - u_q \\ -(U - u_k - u_{k+1}) & V - u_q(U - u_q) - (Y - X_{q-1} - X_q) \end{pmatrix} \\ C_{kq} &= \begin{pmatrix} U - u_k & U - u_q \\ V - u_k(U - u_k) - (Y - X_{k-1} - X_k) & V - u_q(U - u_q) - (Y - X_{q-1} - X_q) \end{pmatrix}. \end{aligned}$$

Using the expression of A_{kq} we deduce that $u_1 + u_2 = u_2 + u_3 = \dots = u_{n-1} + u_n = u_n + u_1$.

The case $n \in 2\mathbb{N} + 1$. In this case we have $u_1 = u_2 = \dots = u_n = u$, and $U = nu$, and $V = \frac{(n-1)n}{2}u^2$ and

$$B_{kq} = \begin{pmatrix} -1 & (n-1)u \\ -(n-2)u & \frac{(n-2)(n-1)}{2}u^2 - (Y - X_q - X_{q-1}) \end{pmatrix}.$$

Using the condition that $\det(B_{kq}) = 0$, we obtain

$$Y - X_q - X_{q-1} = -\frac{(n-2)(n-1)}{2}u^2, \quad \forall q \in \{1, \dots, n\}.$$

Adding these relations we have

$$Y = -\frac{(n-1)n}{2}u^2, \quad \text{and } X_1 = X_2 = \dots = X_n = X \quad \text{and } X = -\frac{n-1}{2}u^2.$$

With this relation we have

$$C_{kq} = \begin{pmatrix} (n-1)u & (n-1)u \\ (n-2)(n-1)u^2 & (n-2)(n-1)u^2 \end{pmatrix}.$$

We observe that the equality $\det(C_{kq}) = 0$ is verified. In conclusion we have

$$M_{(1)}^{\mathbb{I}_{23}} = \left\{ \left(-\frac{n-1}{2}u^2, \dots, -\frac{n-1}{2}u^2, u, \dots, u \right) \mid u \in \mathbb{R} \right\}.$$

The case $n \in 2\mathbb{N}$. In this case $u_1 = u_3 = \dots = u_{n-1}$, $u_2 = u_4 = \dots = u_n$ and we have $U = \frac{n}{2}(u_1 + u_2)$. Using (5.2) we obtain $V = \frac{n(n-2)}{8}(u_1^2 + u_2^2) + \frac{n^2}{4}u_1u_2$. From the relation $\det(B_{kq}) = 0$, we have

$$Y - X_{q-1} - X_q = V - (U - u_q)\left(\frac{n-2}{n}U + u_q\right). \quad (5.9)$$

Consequently, $X_1 + X_2 = X_3 + X_4 = \dots = X_{n-1} + X_n$ and $X_2 + X_3 = X_4 + X_5 = \dots = X_n + X_1$ which further implies that $X_1 = X_3 = \dots = X_{n-1}$, $X_2 = X_4 = \dots = X_n$ and $Y = \frac{n}{2}(X_1 + X_2)$. By substitution into (5.9) we obtain $X_1 + X_2 = -\frac{n}{4}(u_1 + u_2)^2 + u_1u_2$. We have

$$M_{(1)}^{\mathbb{I}_{23}} = \{(X_1, X_2, \dots, X_1, X_2, u_1, u_2, \dots, u_1, u_2) \mid X_1 + X_2 = -\frac{n}{4}(u_1 + u_2)^2 + u_1u_2\}.$$

The study of $M_{(2)}^{\mathbb{I}_{123}}$.

We introduce, for $k, q, r \in \{1, \dots, n\}$, the matrices

$$\begin{aligned} A_{kqr} &= \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & U - u_r \\ -(U - u_k - u_{k+1}) & -(U - u_q - u_{q+1}) & V_r \end{pmatrix} \\ B_{kqr} &= \begin{pmatrix} 0 & 1 & 1 \\ -1 & U - u_q & U - u_r \\ -(U - u_k - u_{k+1}) & V_q & V_r \end{pmatrix} \\ C_{kqr} &= \begin{pmatrix} 1 & 1 & 1 \\ U - u_k & U - u_q & U - u_r \\ V_k & V_q & V_r \end{pmatrix}, \end{aligned}$$

where $V_s = V - u_s(U - u_s) - (Y - X_{s-1} - X_s)$. We observe that $\text{rank}(\nabla \mathbb{I}_{123}) = 2 \Leftrightarrow \det(A_{kqr}) = \det(B_{kqr}) = \det(C_{kqr}) = 0 \quad \forall k, q, r \in \{1, \dots, n\}$. The equations $\det(A_{kqr}) = 0 \quad \forall k, q, r \in \{1, \dots, n\}$ give us

$$u_1 + u_2 = u_2 + u_3 = \dots = u_{n-1} + u_n = u_n + u_1.$$

The case $n \in 2\mathbb{N} + 1$. We have $u := u_1 = \dots = u_n$. With this notation we obtain $U = nu$, and $V = \frac{n(n-1)}{2}u^2$ and $V_k = -\frac{n(n-1)}{2}u^2 - (Y - X_{k-1} - X_k)$. It is easy to see that $\det(C_{kqr}) = 0$. We have the equivalences

$$\det(B_{kqr}) = 0 \Leftrightarrow \det \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & (n-1)u \\ V_k & V_q - V_r & V_r \end{pmatrix} = 0$$

$$\Leftrightarrow V_r = V_q \Leftrightarrow X_{r-1} + X_r = X_{q-1} + X_q.$$

Because $n \in 2\mathbb{N} + 1$, we obtain $X_1 = X_2 = \dots = X_n$. In this case we have

$$M_{(2)}^{\mathbb{I}_{123}} = \{(X, \dots, X, u, \dots, u) \mid X, u \in \mathbb{R}\}.$$

The case $n \in 2\mathbb{N}$. We have $u_1 = u_3 = \dots = u_{n-1}$, $u_2 = u_4 = \dots = u_n$. By calculus we obtain

$$\det(C_{kqr}) = 0 \Leftrightarrow \det \begin{pmatrix} 1 & 0 & 0 \\ U - u_k & u_k - u_q & u_k - u_r \\ V_k & V_q - V_k & V_r - V_k \end{pmatrix} = 0.$$

If $q = k+1$ and $r = k+2$ we obtain $V_k = V_{k+2}$, which implies that $X_1 + X_2 = X_3 + X_4 = \dots = X_{n-1} + X_n$ and $X_2 + X_3 = X_4 + X_5 = \dots = X_n + X_1$. Consequently, we have $X_1 = X_3 = \dots = X_{n-1}$ and $X_2 = X_4 = \dots = X_n$. We observe that if $s - t \in 2\mathbb{Z}$, then $u_s = u_t$ and $V_s = V_t$, which implies that $\det(C_{kqr}) = 0 \ \forall k, q, r \in \{1, \dots, n\}$. For the matrices B_{kqr} we have the following properties:

i) $\det B_{111} = \det B_{112} = 0$ (by calculus).

ii) $\det B_{kqr} = -\det B_{krq}$.

iii) If $q_1 - q_2 \in 2\mathbb{N}$ and $r_1 - r_2 \in 2\mathbb{N}$ then $\det B_{k_1 q_1 r_1} = \det B_{k_2 q_2 r_2}$.

Using these properties we deduce that $\det B_{kqr} = 0$, $\forall k, q, r \in \{1, \dots, n\}$ and we obtain

$$M_{(2)}^{\mathbb{I}_{123}} = \{(X_1, X_2, \dots, X_1, X_2, u_1, u_2, \dots, u_1, u_2) \mid X_1, X_2, u_1, u_2 \in \mathbb{R}\}.$$

The computations for the case of non-periodic lattice.

If we consider the variables $(X_1, \dots, X_{n-1}, u_1, \dots, u_n)$ we obtain

$$\nabla F_1 = (\underbrace{0, \dots, 0}_{n-1}, \underbrace{1, \dots, 1}_n) \quad (5.10)$$

$$\nabla F_2 = (\underbrace{1, \dots, 1}_{n-1}, u_1, \dots, u_n) \quad (5.11)$$

$$\nabla F_3 = (u_1 + u_2, \dots, u_{n-1} + u_n, X_1 + u_1^2, X_2 + X_1 + u_2^2, \dots, X_{n-2} + X_{n-1} + u_{n-1}^2, X_{n-1} + u_n^2). \quad (5.12)$$

The study of $M_{(0)}^{F_3}$.

The elements of $M_{(0)}^{F_3}$ are the solutions of the system

$$\begin{cases} u_1 + u_2 = u_2 + u_3 = \dots = u_{n-1} + u_n = 0 \\ X_1 + u_1^2 = X_2 + X_1 + u_2^2 = \dots = X_{n-2} + X_{n-1} + u_{n-1}^2 = X_{n-1} + u_n^2 = 0. \end{cases}$$

The case $n \in 2\mathbb{N} + 1$. We obtain

$$M_{(0)}^{F_3} = \{(\underbrace{0, \dots, 0}_{n-1}, \underbrace{0, \dots, 0}_n)\}.$$

The case $n \in 2\mathbb{N}$. In this situation $u_1 = u_3 = \dots = u_{n-1} = u$, $u_2 = u_4 = \dots = u_n = -u$ and $X_k = -u^2$ if $k \in 2\mathbb{N} + 1$ and $X_k = 0$ if $k \in 2\mathbb{N}$. We have

$$M_{(0)}^{F_3} = \{(\underbrace{X, 0, \dots, X, 0}_{n-1}, \underbrace{X, u_1, u_2, \dots, u_1, u_2}_n) \mid u_1 + u_2 = 0, X = u_1 u_2\}.$$

The study of $M_{(0)}^{\mathbb{F}_{ij}}$, $M_{(1)}^{\mathbb{F}_{ij}}$ **with** $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$.

The elements of $M_{(1)}^{\mathbb{F}_{13}}$ are the solutions of the system

$$\begin{cases} u_1 + u_2 = u_2 + u_3 = \dots = u_{n-1} + u_n = 0 \\ X_1 + u_1^2 = X_2 + X_1 + u_2^2 = \dots = X_{n-2} + X_{n-1} + u_{n-1}^2 = X_{n-1} + u_n^2. \end{cases}$$

The case $n \in 2\mathbb{N} + 1$. We obtain that

$$M_{(1)}^{\mathbb{F}_{13}} = \{(\underbrace{0, \dots, 0}_{n-1}, \underbrace{0, \dots, 0}_n)\}.$$

The case $n \in 2\mathbb{N}$. In this situation $u_1 = u_3 = \dots = u_{n-1} = u$, $u_2 = u_4 = \dots = u_n = -u$ and $X_k = X$ if $k \in 2\mathbb{N} + 1$ and $X_k = 0$ if $k \in 2\mathbb{N}$. We have

$$M_{(1)}^{\mathbb{F}_{13}} = \{(\underbrace{X, 0, \dots, X, 0}_{n-1}, \underbrace{X, u_1, u_2, \dots, u_1, u_2}_n) \mid u_1 + u_2 = 0\}.$$

For a point of the set $M_{(1)}^{\mathbb{F}_{23}}$ we have

$$\det(A_{kq}) = 0, \quad \det(B_{kq}) = 0, \quad \det(C_{kq}) = 0, \quad \forall k, q \quad (5.13)$$

where

$$A_{kq} = \begin{pmatrix} 1 & 1 \\ u_k + u_{k+1} & u_q + u_{q+1} \end{pmatrix}, \quad k, q \in \{1, \dots, n-1\} \quad (5.14)$$

$$B_{kq} = \begin{pmatrix} 1 & u_q \\ u_k + u_{k+1} & X_{q-1} + X_q + u_q^2 \end{pmatrix}, \quad k \in \{1, \dots, n-1\} \text{ and } q \in \{1, \dots, n\} \quad (5.15)$$

$$C_{kq} = \begin{pmatrix} u_k & u_q \\ X_{k-1} + X_k + u_k^2 & X_{q-1} + X_q + u_q^2 \end{pmatrix}, \quad k, q \in \{1, \dots, n\}. \quad (5.16)$$

Using the expression of A_{kq} we deduce that $u_1 + u_2 = u_2 + u_3 = \dots = u_{n-1} + u_n$ and consequently $u_k = u_q$ if $k - q \in 2\mathbb{Z}$. The matrices B_{kq} have the form

$$B_{kq} = \begin{pmatrix} 1 & u_1 \\ u_1 + u_2 & X_{q-1} + X_q + u_1^2 \end{pmatrix} \quad \text{if } q \in 2\mathbb{N} + 1$$

and

$$B_{kq} = \begin{pmatrix} 1 & u_2 \\ u_1 + u_2 & X_{q-1} + X_q + u_2^2 \end{pmatrix} \quad \text{if } q \in 2\mathbb{N}.$$

We have $\det B_{kq} = 0$ if and only if $X_1 = X_1 + X_2 = \dots = X_{n-2} + X_{n-1} = X_{n-1} = u_1 u_2$.

The case $n \in 2\mathbb{N} + 1$. It is easy to see that $X_1 = X_2 = \dots = X_{n-1} = 0$ and $u_1 u_2 = 0$. We observe that

$$M_{(1)}^{\mathbb{F}_{23}} = \{(\underbrace{0, \dots, 0}_{n-1}, \underbrace{u_1, 0, \dots, u_1, 0, u_1}_n) \mid u_1 \in \mathbb{R}\} \cup \{(\underbrace{0, \dots, 0}_{n-1}, \underbrace{0, u_2, \dots, 0, u_2, 0}_n) \mid u_2 \in \mathbb{R}\}.$$

The case $n \in 2\mathbb{N}$. We deduce that $X_1 = X_3 = \dots = X_{n-1} = X$, $X_2 = X_4 = \dots = X_{n-2} = 0$ and $u_1 u_2 = X$. In this case we have

$$M_{(1)}^{\mathbb{F}_{23}} = \{(\underbrace{X, 0, X, \dots, 0}_{n-1}, \underbrace{X, u_1, u_2, \dots, u_1, u_2}_n) \mid X = u_1 u_2\}.$$

The study of $M_{(2)}^{\mathbb{F}_{123}}$.

For a point of the set $M_{(2)}^{\mathbb{F}_{123}}$ we have

$$\det(A_{kqr}) = 0, \quad \det(B_{kqr}) = 0, \quad \det(C_{kqr}) = 0, \quad \forall k, q, r \quad (5.17)$$

where

$$A_{kqr} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & u_r \\ u_k + u_{k+1} & u_q + u_{q+1} & X_{r-1} + X_r + u_r^2 \end{pmatrix}, \quad k, q \in \{1, \dots, n-1\}, \quad r \in \{1, \dots, n\} \quad (5.18)$$

$$B_{kqr} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & u_q & u_r \\ u_k + u_{k+1} & X_{q-1} + X_q + u_q^2 & X_{r-1} + X_r + u_r^2 \end{pmatrix}, \quad k \in \{1, \dots, n-1\}, q, r \in \{1, \dots, n\} \quad (5.19)$$

$$C_{kqr} = \begin{pmatrix} 1 & 1 & 1 \\ u_k & u_q & u_r \\ X_{k-1} + X_k + u_k^2 & X_{q-1} + X_q + u_q^2 & X_{r-1} + X_r + u_r^2 \end{pmatrix}, \quad k, q, r \in \{1, \dots, n\} \quad (5.20)$$

Using the expression of A_{kqr} we deduce that $u_1 + u_2 = u_2 + u_3 = \dots = u_{n-1} + u_n$ and consequently $u_k = u_q$ if $k - q \in 2\mathbb{Z}$. For $k \in \{1, \dots, n-1\}$, we have

$$\det B_{k,k+1,k} = \det \begin{pmatrix} 0 & 0 & 1 \\ 1 & u_{k+1} - u_k & u_k \\ u_k + u_{k+1} & X_{k+1} - X_{k-1} + u_{k+1}^2 - u_k^2 & X_{k-1} + X_k + u_k^2 \end{pmatrix} = 0$$

and we deduce that $X_{k+1} = X_{k-1}$.

The case $n \in 2\mathbb{N} + 1$. It is easy to see that $X_1 = X_2 = \dots = X_{n-1} = 0$, $u_1 = u_3 = \dots = u_n$ and $u_2 = u_4 = \dots = u_{n-1}$. All the conditions (5.17) are verified and the set is

$$M_{(2)}^{\mathbb{F}_{123}} = \{(\underbrace{0, \dots, 0}_{n-1}, \underbrace{u_1, u_2, \dots, u_1}_n) \mid u_1, u_2 \in \mathbb{R}\}.$$

The case $n \in 2\mathbb{N}$. It is easy to see that $X_1 = X_3 = \dots = X_{n-1} = X$, $X_2 = X_4 = \dots = X_{n-2} = 0$, $u_1 = u_3 = \dots = u_{n-1}$ and $u_2 = u_4 = \dots = u_n$. All the conditions (5.17) are verified and the set is

$$M_{(2)}^{\mathbb{F}_{123}} = \{(\underbrace{X, 0, \dots, X, 0, X}_{n-1}, \underbrace{u_1, u_2, \dots, u_1, u_2}_n) \mid X, u_1, u_2 \in \mathbb{R}\}.$$

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